Mathematical Programming Formulations of Transportation Models

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***Abstract* - The next generation of transportation, location, models will most probably emerge from mathematical programming formulations. Presented are simple numerical examples of trip assignment and population location, both described as optimization problems, in mathematical programming formulations. A trip assignment model with constant link costs less described first, and then the same model is modified to show the consequences of a How-dependent link cost formulation. In similar fashion, a linear model of least cost population location is transformed into a nonlinear model that incorporates dispersion of location due to differences in locators' preferences or perceptions. It less then showed how the trip assignment model and the location model can be combined into a single nonlinear programming formulation that solves both problems simultaneously.**

***Index Terms* - Transportation problem, Bottleneck Transportation Problem, Mathematical Programming**

INTRODUCTION

Mathematical programming is one of the most important techniques avail-able for quantitative decision making. The general purpose of mathematical programming is finding an optimal solution for allocation of limited re-sources to perform competing activities. The optimality is defined with respect to important performance evaluation criteria, such as cost, time, and profit. Mathematical programming uses a compact mathematical model for describing the problem of concern. The solution is searched among all feasible alternatives. The search is executed in an intelligent manner, allowing the evaluation of problems with a large number of feasible solutions. Mathematical programming finds many applications in supply chain management, at all decision-making levels. It is also widely used for supply chain configuration purposes. Out of several classes of mathematical programming models, mixed-integer programming models are used most frequently. Other types of models, such as stochastic and multi-objective programming models, are also emerging to handle more complex supply chain configuration problems. Although these models are often more appropriate, computational complexity remains an important issue in the application of mathematical programming models for supply chain configuration.

There has been considerable refinement of practical methods of forecasting urban location and transportation patterns during the past 10 to 15 years. Although there is continuing discussion and development, and even the best of forecasts are far from perfect, there appears to be a greater consensus on what methods are clearly outmoded and in what directions future efforts should move. Among the most sophisticated practical methods of transportation are the extended spatial interaction models, especially when they are included in comprehensive integrated model systems.

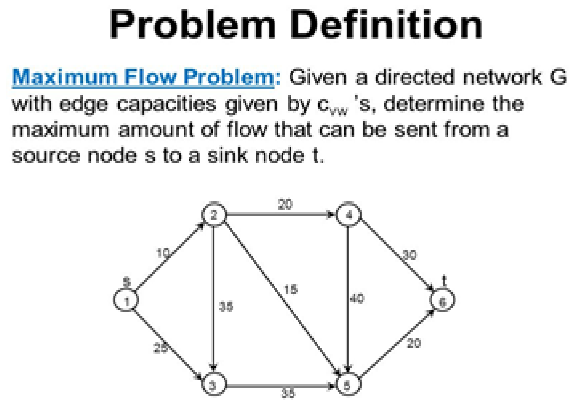
In addition to these practical developments there have also been important theoretical developments. On the transportation side these include the development of discrete choice models, especially for travel demand and mode choice (Ben-Akiva and Lerman, 1985), and the development of mathematical programming formulations of the traffic assignment problem (Sheffi, 1985). On the location side the development of utility theory as a basis for location models (Anas, 1982) and the general discussion of mathematical programming models as alternate or underlying structures for spatial interaction models (Wilson et al., 1981) were major developments. Some of these developments are important principally because they provide an improved underpinning of existing practical methods; some have shown the existence of clear errors in prior practice; and others may offer substantial improvements for future applications.

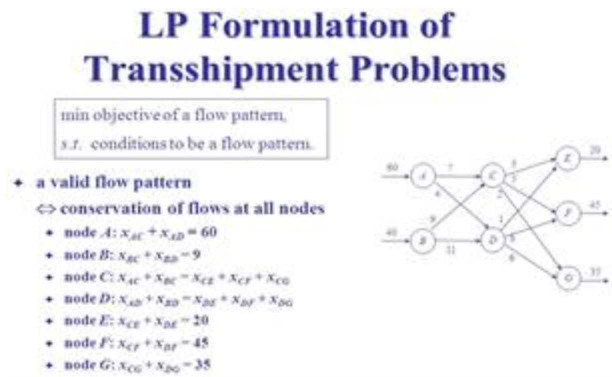
Past experience suggests that there is a lag of 10 years, sometimes more, between the initial development and subsequent practical application of new techniques in transportation and land use forecasting. Thus, although there have been some attempted applications of these methods (Prastacos, 1985), they are far from being the accepted norm. The purpose of this paper is to present some illustrations of the mathematical programming formulations along with some simple numerical examples. The intent is to show some of the benefits, both practical and theoretical, of these formulations and to provide the practical planner with an introduction to this promising new area.

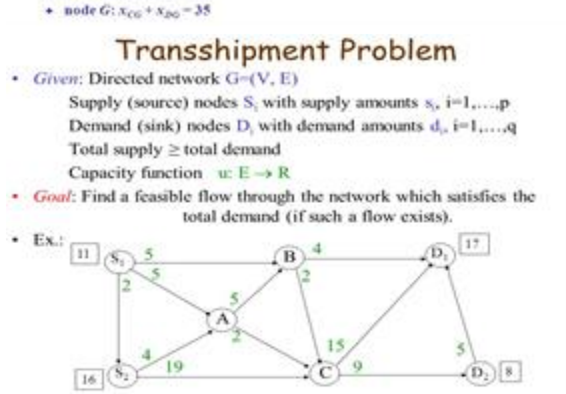
Mathematical programming models are used to optimize decisions concerning execution of certain activities subject to resource constraints. Mathematical programming models have a well-defined structure. They consist of mathematical expressions representing objective function and constraints. The expressions involve parameters and decision variables. The parameters are input data, while the decision variables represent the optimization outcome. The objective function represents modeling objectives and makes some decisions more preferable than others. The constraints limit the values that decision variables can assume.

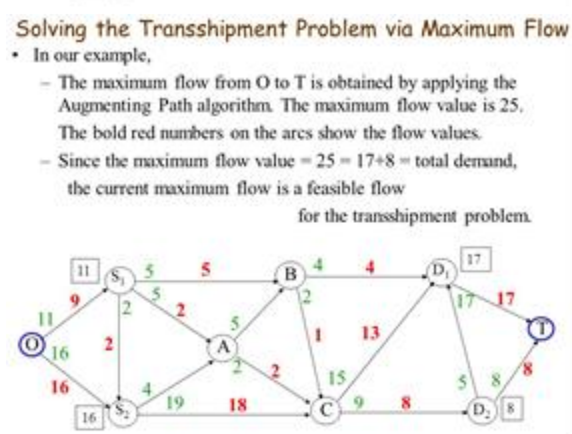
The main advantages of mathematical programming models are that they provide a relatively simple and compact approximation of complex decision-making problems, an ability to efficiently find an optimal set of decisions among a large number of alternatives, and supporting analysis of decisions made. Specifically, in the supply chain configuration problem context, mathematical programming models are excellent for modeling its special aspects. There are also some important limitations. Mathematical programming models have a lower level of validity compared to some other types of models —particularly, simulation. In the supply chain configuration context, mathematical programming models have difficulties representing the dynamic and stochastic aspects of the problem. Additionally, solving of many supply chain configuration problems is computationally challenging.

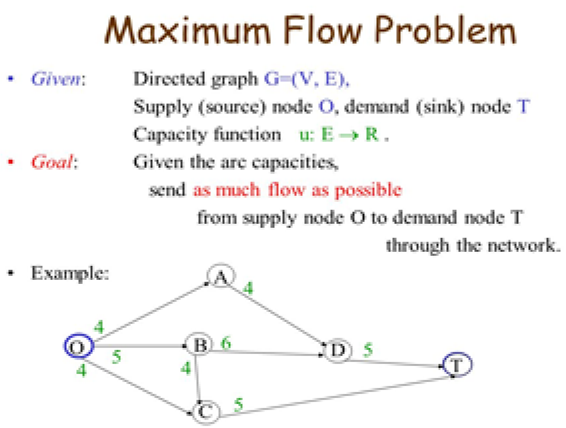
Mathematical programming uses a compact mathematical model for describing the problem of concern. The solution is searched among all feasible alternatives. The search is executed in an intelligent manner, allowing the evaluation of problems with a large number of feasible solutions.

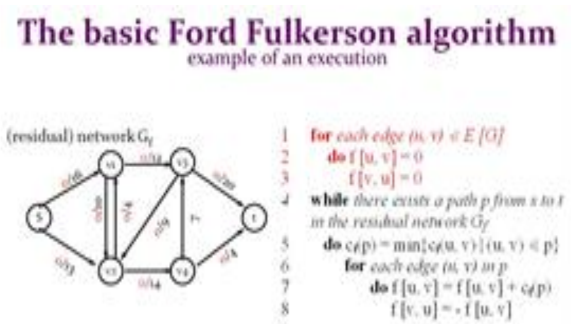


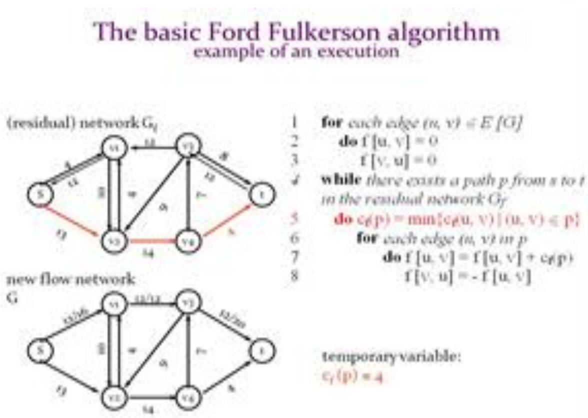







Maximum Flow Problem
• Given: Directed graph G=(V, E),
Supply (source) node O, demand (sink) node T
Capacity function u: E → R .
• Goal: Given the arc capacities,
send as much flow as possible
from supply node O to demand node T
through the network.
• Example:


The basic Ford Fulkerson algorithm
example of an execution
(residual) network G_f
1 for each edge (u, v) ∈ E [G]
2 do f [u, v] = 0
3 f [v, u] = 0
4 while there exists a path p from s to t in the residual network G_f
5 do $c_f(p)$ = min{$c_f(u, v)$ | (u, v) ∈ p}
6 for each edge (u, v) in p
7 do f [u, v] = f [u, v] + $c_f(p)$
8 f [v, u] = - f [u, v]


The basic Ford Fulkerson algorithm
example of an execution
(residual) network G_f
new flow network
G
1 for each edge (u, v) ∈ E [G]
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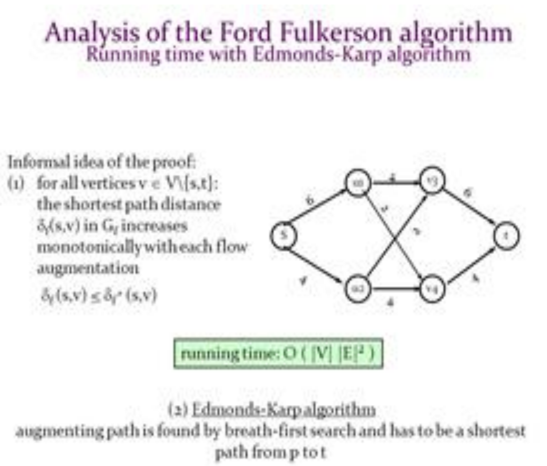
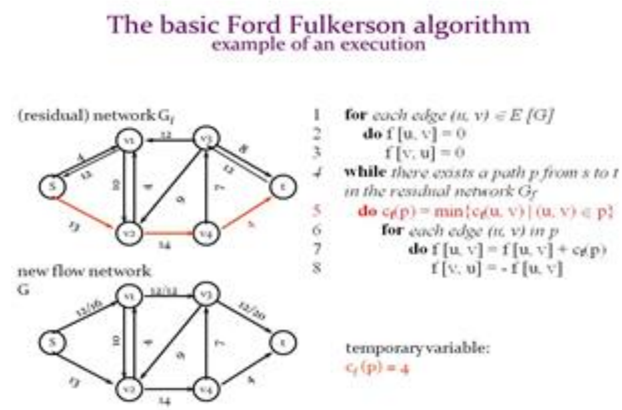
7 do f [u, v] = f [u, v] + $c_f(p)$
8 f [v, u] = - f [u, v]
temporary variable:
$c_f(p) = 4$


Analysis of the Ford Fulkerson algorithm
Running time with Edmonds-Karp algorithm
Informal idea of the proof:
(1) for all vertices v ∈ V\[s,t]:
the shortest path distance $\delta_f(s,v)$ in G_f increases monotonically with each flow augmentation
$\delta_f(s,v) \le \delta_{f'}(s,v)$
running time: O (|V| |E|²)
(2) Edmonds-Karp algorithm
augmenting path is found by breath-first search and has to be a shortest path from p to t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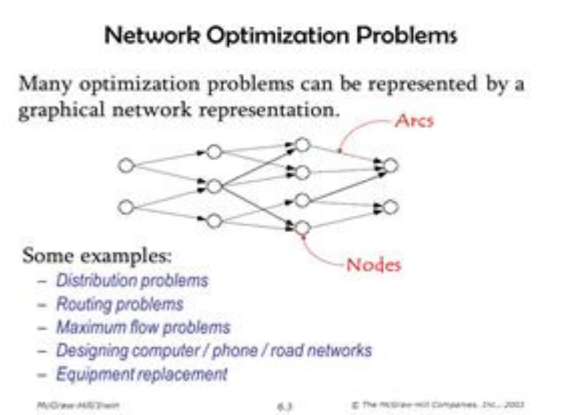
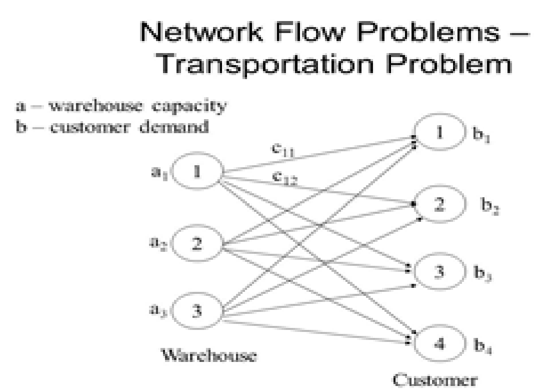
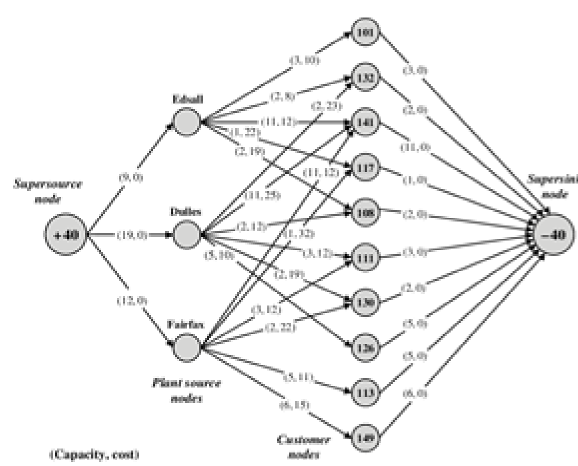
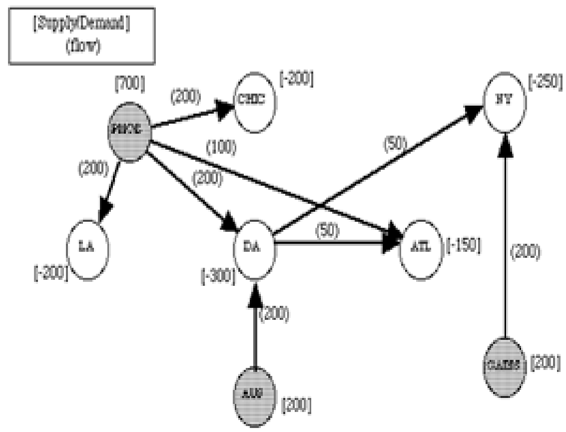

Network Optimization Problems
Many optimization problems can be represented by a graphical network representation.
Arcs
Nodes
Some examples:
– Distribution problems
– Routing problems
– Maximum flow problems
– Designing computer / phone / road networks
– Equipment replacement


Network Flow Problems – Transportation Problem
a – warehouse capacity
b – customer demand
Warehouse
Customer


Supersource node
Supersink node
Edsall
Dulles
Fairfax
Plant source nodes
Customer nodes
(Capacity, cost)



Transportation model

The transportation problem is one of the subclasses of linear programming problem where the objective is to transport various quantities of a single homogeneous product that are initially stored at various origins, to different destinations in such a way that the total transportation is minimum.

Purpose of Transportation models

Transportation models or problems are primarily concerned with the optimal (best possible) way in which a product produced at different factories or plants (called supply origins) can be transported to a number of warehouses (called demand destinations). The objective in a transportation problem is to fully satisfy the destination requirements within the operating production capacity constraints at the minimum possible cost.

Transportation Problem: Characteristics

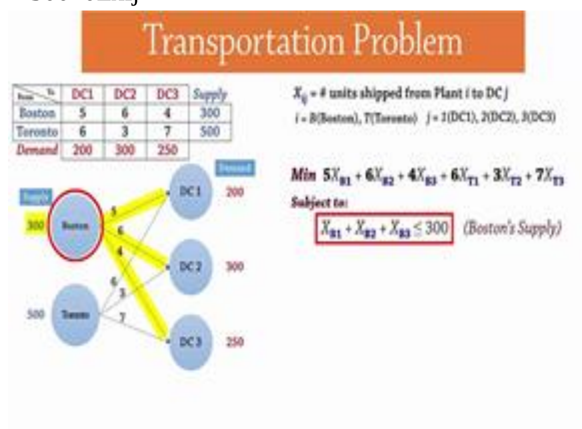
A transportation problem aims to find the best way to fulfill the demand of n demand points using the capacities of m supply points. A product is transported from a number of sources to a number of destinations at the minimum possible cost. Each source is able to supply a fixed number of units of the product, and each destination has a fixed demand for the product. The linear programming model has constraints for supply at each source and demand at each destination. In a balanced transportation model supply equals demand.

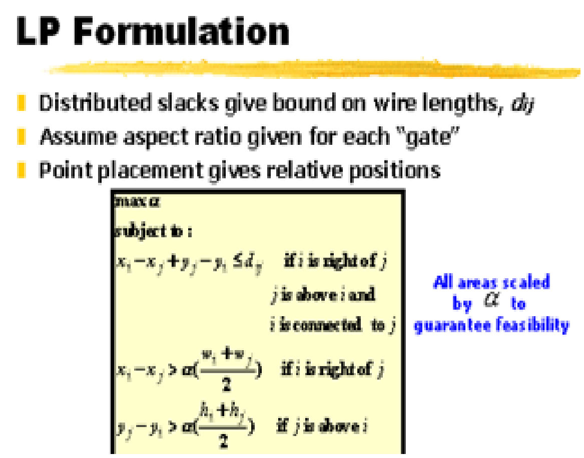
Types of Transportation Problem

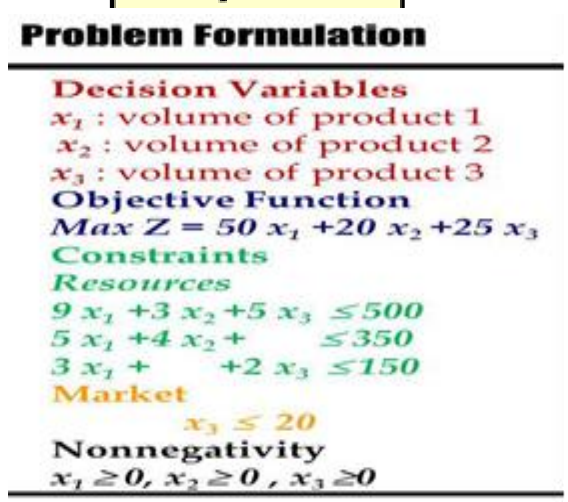
Balanced Transportation Problem Unbalanced Transportation Problem Less Supply as Compared to Demand Less Demand as Compared to Supply

Transportation Model Example Linear Programming Model Formulation

xij = tons of wheat from each grain elevator, i, i = 1, 2, 3, to each mill j = A,B,C Minimize Z = $6x1A + 8x1B + 10x1C + 7x2A + 11x2B + 11x2C x3A + 5x3B + 12x3C subject to: x1A + x1B + x1C = 150 x2A + x2B + x2C = 175 x3A + x3B + x3C = 275 x1A + x2A + x3A = 200 x1B + x2B + x3B = 100 x1C + x2C + x3C = 300 0≥xij







Transportation Model Example
Linear Programming Model Formulation

x_{ij} = tons of wheat from each grain elevator, i, i = 1, 2, 3, to each mill j = A,B,C

Minimize $Z = \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C}$

subject to:

$$x_{1A} + x_{1B} + x_{1C} = 150$$
$$x_{2A} + x_{2B} + x_{2C} = 175$$
$$x_{3A} + x_{3B} + x_{3C} = 275$$
$$x_{1A} + x_{2A} + x_{3A} = 200$$
$$x_{1B} + x_{2B} + x_{3B} = 100$$
$$x_{1C} + x_{2C} + x_{3C} = 300$$
$$x_{ij} \geq 0$$

Summary of the Transportation Simplex Method

The transportation simplex method uses linear programming to solve transportation problems. The goal is to create the optimal solution when there are multiple suppliers and multiple destinations. The data required includes the unit shipping costs, how much each supplier can produce, and how much each destination needs.

For a min problem

Step1. Balance problem.

Step2. Find a bfs. (initial solution)

Step3. Let $u_1=0$, $c_{ij}=u_i+v_j$ for all BV. Find u_i ,v_j.

Step4. Let $\bar{c}_{ij} = u_i+v_j-c_{ij}$

If $\bar{c}_{ij} \leq 0$ for all NBV,

then the current bfs is optimal

else the var with the most positive $\bar{c}_{ij}$ is a new bfs.

Step5. using the new bfs return to step 3 and 4.

For a max problem

Step4'. $\bar{c}_{ij} \geq 0$

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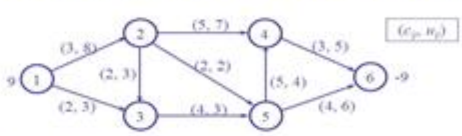
Minimum Cost Flow Models



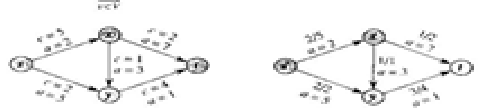
- $N = \{1, 2, 3, 4, 5, 6\}$
- $A = \{(1, 2), (1, 3), (2, 3), (2, 4), (2, 5), (3, 5), (4, 6), (5, 4), (5, 6)\}$
- $b(1) = 9$, $b(6) = -9$, and $b(i) = 0$ for $i = 2$ to 5

Minimum Cost Flow

minimize $\sum_{(u,v) \in E} a(u,v) f(u,v)$

subject to

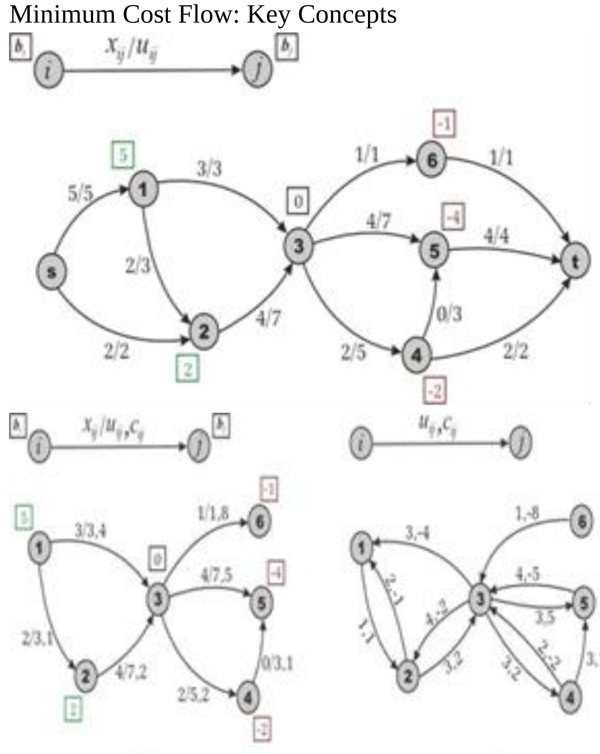
$$f(u,v) \leq c(u,v) \quad \text{for each } u, v \in V,$$
$$f(u,v) = -f(v,u) \quad \text{for each } u, v \in V,$$
$$\sum_{w \in V} f(u,w) = 0 \quad \text{for each } u \in V - \{s,t\},$$
$$\sum_{w \in V} f(s,w) = d.$$



(a) (b)

The minimum-cost flow problem (MCFP) is an optimization and decision problem to find the cheapest possible way of sending a certain amount of flow through a flow network. A typical application of this problem involves finding the best delivery route from a factory to a warehouse where the road network has some capacity and cost associated. The minimum cost flow problem is one of the most fundamental among all flow and circulation problems because most other such problems can be cast as a minimum cost flow problem and also that it can be solved efficiently using the network simplex algorithm.

Minimum Cost Flow: Key Concepts



(a) (b)

How do you solve minimum cost problems?

Minimum weight bipartite matching

The idea is to reduce this problem to a network flow problem. Let G′ = (V′ = A ∪ B, E′ = E). Assign the capacity of all the edges in E′ to 1. Add a source vertex s and connect it to all the vertices in A′ and add a sink vertex t and connect all vertices inside group B′ to this vertex.

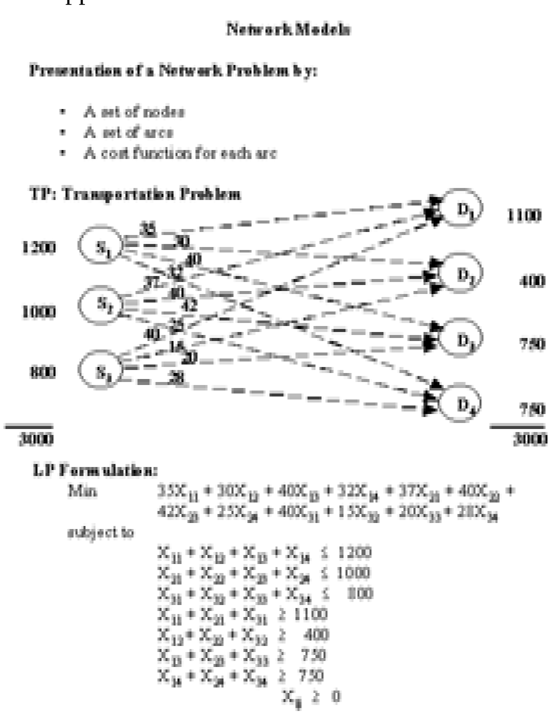
Which algorithm is used to solve a minimum flow problem?

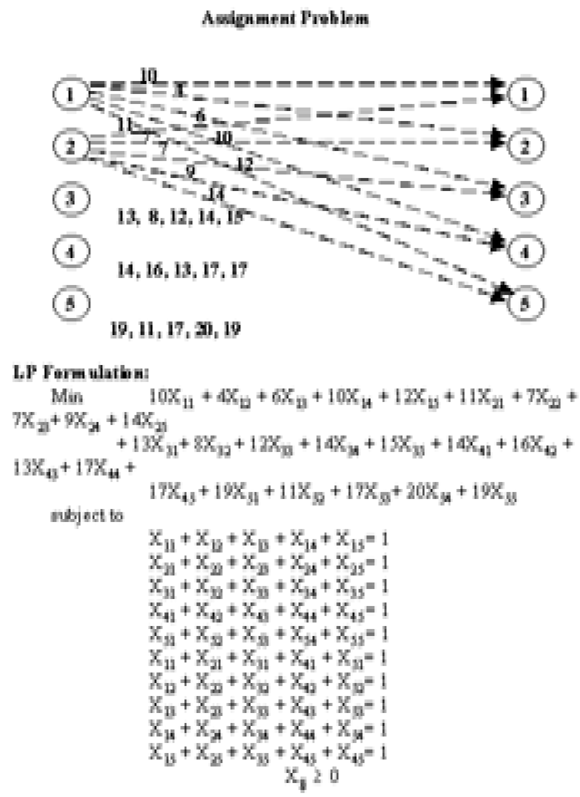
Cycle Cancelling Algorithm:

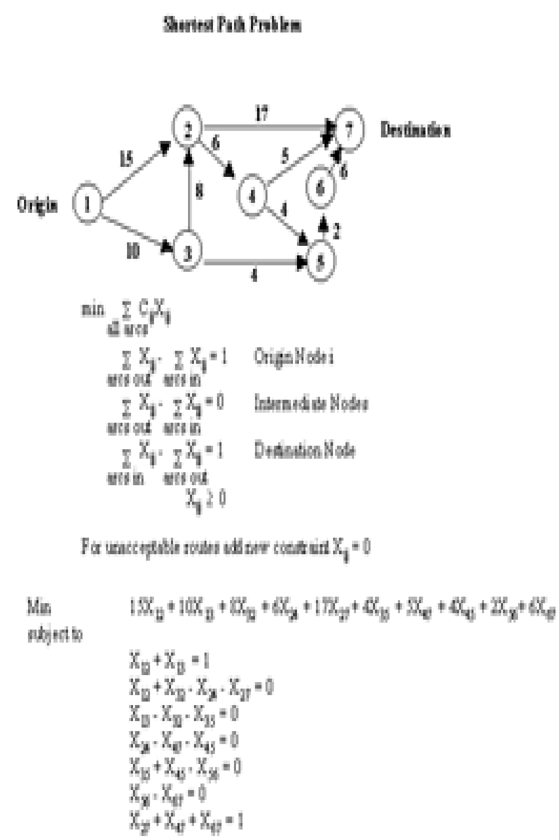
This algorithm is used to find min cost flow along the flow network. Pseudo code for this algorithm is provided below. Negative cycle in cost network is cycle with sum of costs of all the edges in the cycle is negative.

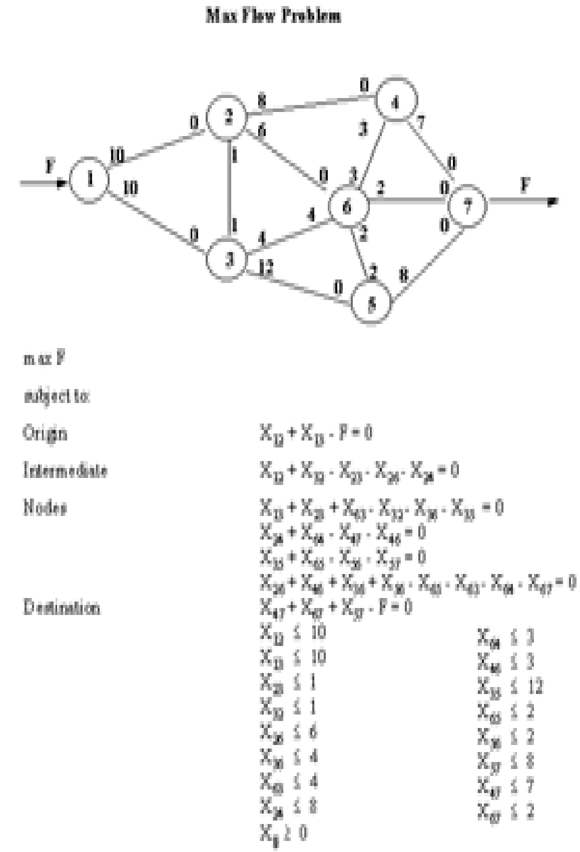
How do you solve maximum flow problem?
It is defined as the maximum amount of flow that the network would allow to flow from source to sink. Multiple algorithms exist in solving the maximum flow problem. Two major algorithms to solve these kinds of problems are Ford-Fulkerson algorithm and Dinic's Algorithm.

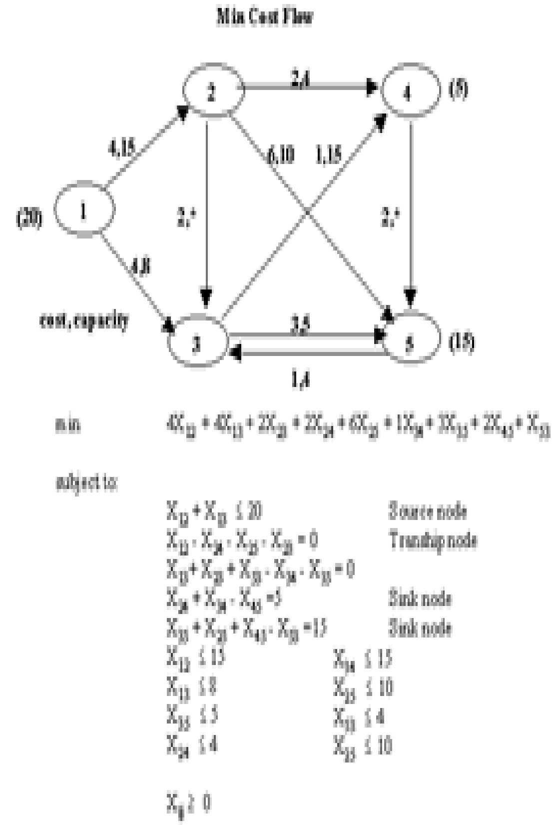
Integer Optimization and the Network Models
Network models and integer programs are applicable for an enormous known variety of decision problems. Some of these decision problems are really physical problems, such as transportation or flow of commodities. Many network problems are more of abstract representations of processes or activities, such as the critical path activity network in project management. These problems are easily illustrated by using a network of arcs, and nodes. Standard linear program assumes that decision variables are continuous. However, in many applications, fractional values may be of little use as shown in some presented useful applications.

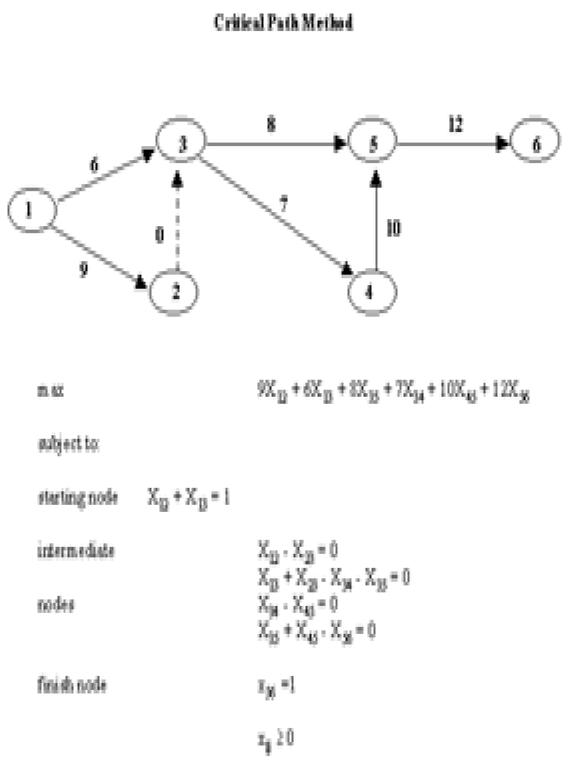












CONCLUSIONS

Numerous model tests using the kinds of models described here indicate that linear mathematical programming models of location are inherently idealistic. The least-cost zone will get all possible locators even if the next-to least-cost zone is only marginally more expensive. The objective function component weighting problem implies that an arbitrary difference in units of measurement can result in one component of a model solution's being dominant over another.

Perhaps the most important is that developing these model formulations and then testing their behavior gives wonderful insight into various hypotheses. The effects, and general importance, of constraints in such formulations became clearly evident in these experiments. At the same time the experiments clearly illustrated the need for inclusion of dispersion terms in such models. The inference is that although in principle, an optimizing process, in actuality there are obviously other factors that result in a dispersion of a simple "least-cost" optimum. Yet the optimizing process provides a model-building rationale that can be particularly helpful in understanding the implications of model structure and can thus, in turn, be expected to improve modeling practice as well.

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